Coherence enhancement in nonlinear systems subject to multiplicative Ornstein-Uhlenbeck noise

Juan L. Cabrera*
Department of Neurology, MC-2030, The University of Chicago, 5841 South Maryland Avenue, Chicago, Illinois 60637 and Laboratorio de Física Estadística de Sistemas Desordenados, Centro de Física, IVIC, Apartado 21827, Caracas 1020A, Venezuela

J. Gorroñogoitia and F. J. de la Rubia
Departamento de Física Fundamental, Universidad Nacional de Educación a Distancia, Apartado Postal 60141, 28080 Madrid, Spain
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We show that for two biologically relevant models with self-sustained oscillations under the action of a multiplicative Ornstein-Uhlenbeck process, their coherence response behaves nonmonotonically with the process correlation time. There is a correlation time for which the quality factor is optimized. This phenomenon is a consequence of the interplay between the correlation time and the system's periodicity. This relation is evidenced through a power law relation with an exponent close to $-\frac{1}{2}$.

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I. INTRODUCTION

Noise is ubiquitous. Over the past decades nonlinear systems with noise have attracted the attention of many researchers from different scientific areas. The reason is the constructive role noise plays in processes involving self-organization, pattern formation, and coherence, as well as in biological information processing, energy transduction, and functionality. A main example is stochastic resonance (SR), a phenomenon by which the addition of noise to a system can enhance its coherent response [1,2] (see Ref. [3] for an extensive review). In its most popular assertion SR is associated with a subthreshold nonlinear system, an additive white noise and a periodic external force. Nevertheless, SR can also occur in systems with different characteristics, e.g., in systems with external aperiodic forcing [4] and in autonomous systems without external periodic force but with intrinsic oscillatory behavior [5]. It also happens in multiplicative stochastic systems [6] or in systems perturbed by colored noise [7,8]. It can even be found in arrays of oscillators [9] or in simple systems with time delay [10]. Only very recently the situation in which the system is subject to both multiplicative and colored noise has been discussed in the literature. In Refs. [11–13] the authors analyze the influence of multiplicative colored noise on periodically driven linear systems, discussing the appearance of nonmonotonous responses by changing either the noise intensity or the correlation time. Moreover, evidence of an SR response in nonlinear systems with multiplicative colored noise without periodic forcing but with self-sustained oscillations has been also reported [14]. These works extend the SR phenomena to systems in which the enhancement response appears as a function of the noise correlation time. This behavior could be properly named correlation time SR, in short τSR.

II. THE MODELS

In this brief note we analyze the coherence response of two nonlinear systems with self-sustained oscillations when the noise correlation varies. We considered the Sel'kov model for the glycolytic oscillator [15]

$$\dot{x} = -x + \lambda y + x^2 y,$$
$$\dot{y} = b - \lambda y - x^2 y, \qquad (1)$$

and the Odell model from population dynamics [16]

$$\dot{x} = -x[x(1-x)-y],$$
$$\dot{y} = y(x-\lambda). \qquad (2)$$

Both models exhibit a supercritical Hopf bifurcation at a certain value of the control parameter $\lambda \equiv \lambda_H$. (In the Sel'kov model the bifurcation point depends also on the value of b). The stochastic versions of these dynamics are obtained by substituting the deterministic control parameter, λ, by a time dependent parameter

$$\lambda_t = \lambda + \zeta_t, \qquad (3)$$

where ζ_t is an stochastic perturbation [17]. This perturbation is assumed to be an Ornstein-Uhlenbeck process (OU), i.e., a stationary Gaussian Markov noise with zero mean,

$$\langle \zeta_t \rangle = 0, \qquad (4)$$

and exponential correlation

$$\langle \zeta_t \zeta_{t'} \rangle = \frac{D}{\tau} \exp(-|t-t'|/\tau), \qquad (5)$$

where τ is the correlation time, and $D/\tau = \sigma^2$ is the variance of the noise. In the following we will refer to the noise intensity as σ.

The numerical integration has been carried out with $\langle \lambda_t \rangle = \lambda$ in the limit cycle parameter domain with an order two explicit weak scheme [18] and with a second-order strong scheme [19].

*Author to whom correspondence must be addressed. Email address: jlc@ace.bsd.uchicago.edu

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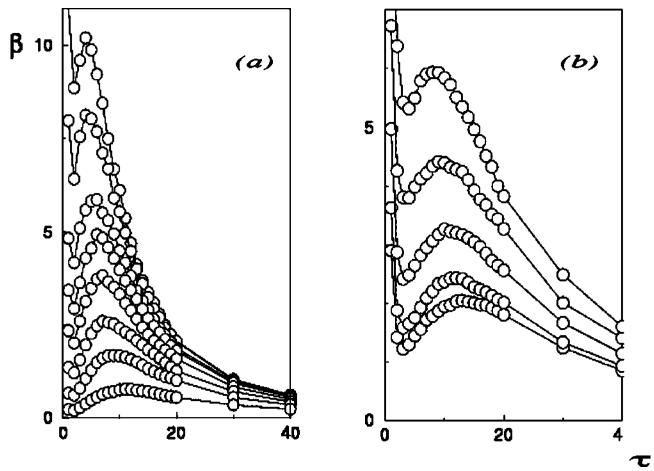

FIG. 1. Quality factor for (a) the Sel'kov model evaluated with $b=0.6$ and different values of $\langle\lambda_t\rangle=0.102$, 0.106, 0.110, 0.112, 0.114, 0.116, 0.118, and 0.120 from top to bottom; $\sigma=5\times10^{-3}$ and (b) the Odell model for different values of $\langle\lambda_t\rangle=0.4960$, 0.4955, 0.4945, 0.4935, and 0.4925 from top to bottom; $\sigma=10^{-3}$. In both cases we used 100 realizations and a time step $\Delta t=10^{-3}$.

III. A POWER LAW TIME COUPLING

To analyze the coherence enhancement we use the quality factor, β, of the power spectrum, defined by the relation [5] $\beta=h\omega_p/W$, where h is the height of the spectrum principal peak, ω_p is the principal peak frequency, and W is the width of the spectrum principal peak at the height $h/\sqrt{e}$.

In Fig. 1 we show the results of calculating β for $x(t)$ for both Eqs. (1) and (2). It is observed that the coherence reaches a maximum for a particular value of $\tau\sim\tau_r$. For τ larger than τ_r the quality factor decreases monotonically. These figures shows that a random perturbation of fixed intensity acting on the parameter can improve the system coherence if it is exponentially correlated and fluctuates with the optimal τ. The results also indicate that the maximum position depends on $\langle\lambda_t\rangle$.

It is known that in other dynamical systems the periodicity of the limit cycle is related with τ_r [14]. Looking for a similar relationship we start considering that

$$T(\lambda)-T_H\sim\Delta\lambda \tag{6}$$

with T_H being the period at the bifurcation point and $\Delta\lambda=|\lambda-\lambda_H|$ being the parameter distance to the bifurcation point.

However, it is known that the bifurcation point is going to be displaced because of the parametric forcing [20,21]. So the temporal distance (6) should be properly corrected taking into account that displacement. This is achieved rewriting Eq. (6) as

$$\Delta T^*\equiv|T^*(\lambda)-T_H|, \tag{7}$$

where T^* is the period of the deterministic time series of the dynamical system evaluated at a distance from the bifurcation point $\Delta\lambda^*=|\lambda-\lambda_H^*|$. Finally we represent the calculated values for $\tau_r(\Delta\lambda)$ as a function of the effective period

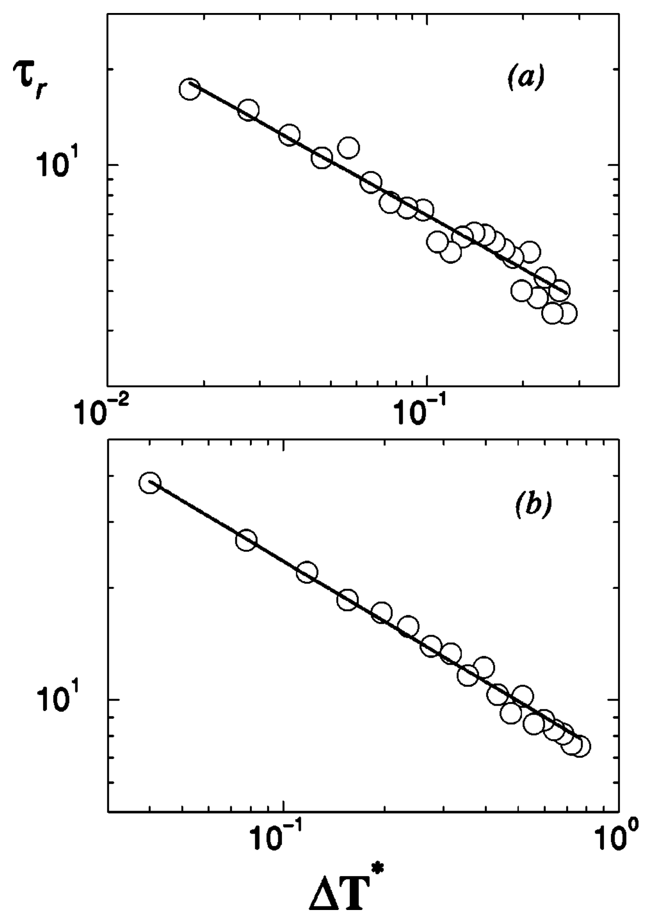

FIG. 2. Resonant correlation time τ_r as a function of the effective period distance ΔT^*. Data for (a) the Sel'kov model evaluated with $b=0.6$, $\langle\lambda_t\rangle$ in the interval [0.100,0.123] taken with increments $\Delta\lambda=10^{-3}$, $\sigma=5\times10^{-3}$ and 50 realizations; and (b) the Odell model evaluated with $\langle\lambda_t\rangle$ in the interval [0.0005,0.0095] taken with increments $\Delta\lambda=5\times10^{-4}$, $\sigma=10^{-3}$ and 50 realizations.

distance ΔT^* in Figs. 2(a) and 2(b) for Eqs. (1) and (2), respectively. It is clear that τ_r obeys the following power law:

$$\tau_r\sim\Delta T(\lambda)^{\gamma_S},$$

$$\tau_r\sim\Delta T(\lambda)^{\gamma_O} \tag{8}$$

with characteristic exponents $\gamma_S\sim-0.56$ and $\gamma_O\sim-0.54$ for the Sel'kov and Odell model, respectively. Now we apply the relation (6) to the former cases obtaining

$$\tau_r\sim T^{-\gamma_S},$$

$$\tau_r\sim T^{-\gamma_O}. \tag{9}$$

In spite of the different dynamics, both exponents are close to the value previously obtained for other systems [14]. This suggests that the phenomenon of coherent enhancement with an OU noise is well characterized by a unique exponent close to $\gamma\sim\frac{1}{2}$.

IV. CONCLUSIONS

Summing up, this work shows that under a parametric OU perturbation an enhancement in the coherence of the output of the Sel'kov and Odell models occurs as a function of τ. This behavior is a consequence of the interplay between two different time scales: the correlation time and the system periodicity. This interplay is evidenced by a power law relation characterized by a unique exponent close to $-\frac{1}{2}$.

Nonmonotonous responses of nonlinear dynamical systems as a function of τ have been reported in other contexts [22,23] or are just appearing in the literature in problems as coupled chaotic oscillators [24]. In Ref. [25] the synchronization response of an array of chaotic elements perturbed by a multiplicative OU noise is analyzed. A power law relating an optimal τ with a relevant system time scale was found there. In the particular case of zero coupling the exponent converges to our predicted value of $-\frac{1}{2}$. This result reinforces the idea regarding the universality of the exponent γ in characterizing the enhancement effect [26].

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